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Signal processing

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Presentation layout

- Fourier series and Fourier transforms
- Leakage
- Aliasing
- Analog versus digital signals
- Digital Fourier transform - FFT
- Concluding remarks

FOURIER ANALYSIS

To pass from the time to the frequency domain, a Fourier transformation is necessary:

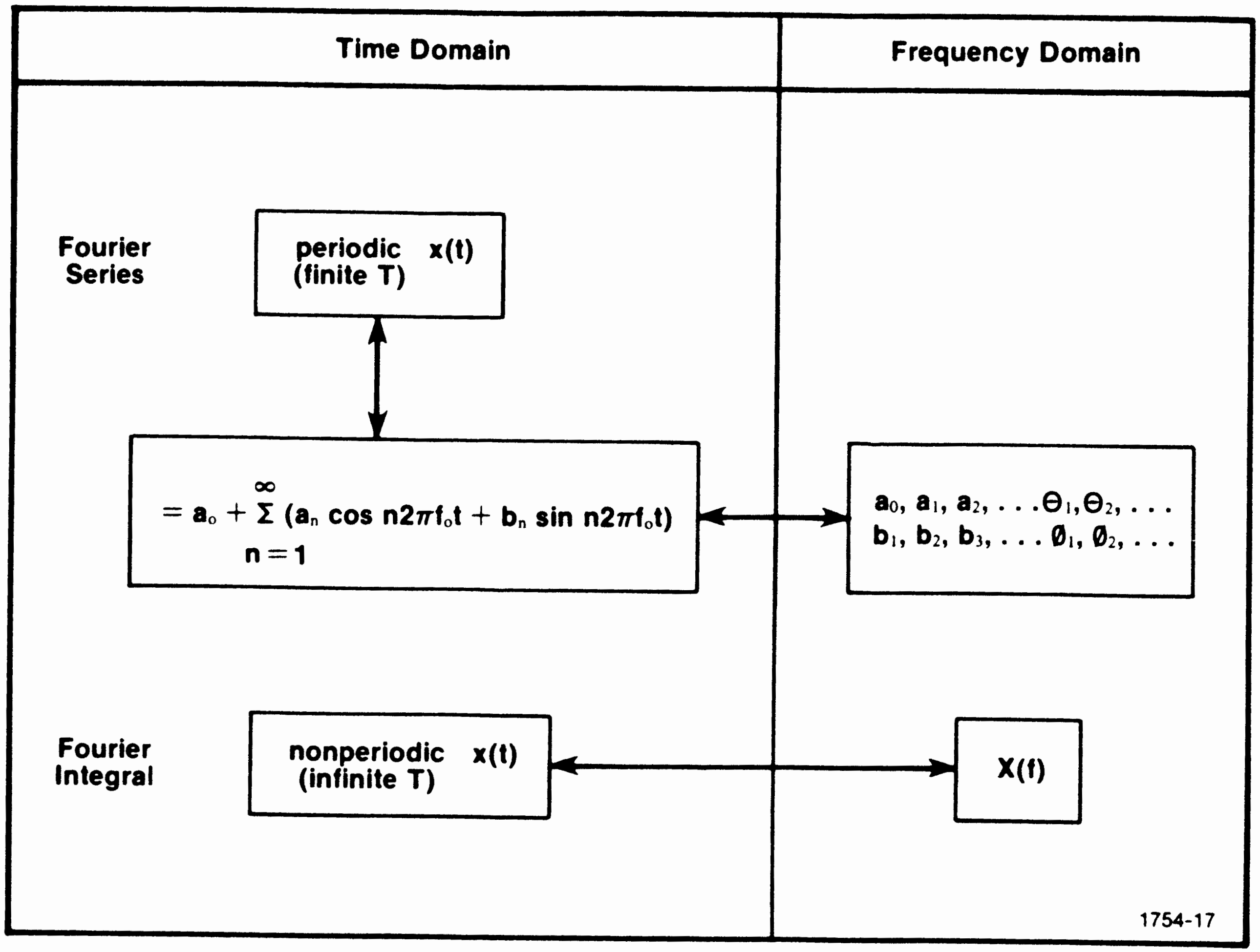
- Fourier series for periodic signals
- Fourier transform for transient signals
- Power spectral density for random signals

$$f(t) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\} \quad ; \quad f(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad c_n = \int_0^T f(t) e^{-jn\omega_0 t} dt$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

$$S_{ff}(\omega) = \lim_{T\to\infty} \frac{F_T(\omega) F_T^*(\omega)}{T}$$

FOURIER ANALYSIS



SOME IMPORTANT PROPERTIES OF F.TRANSFORMS

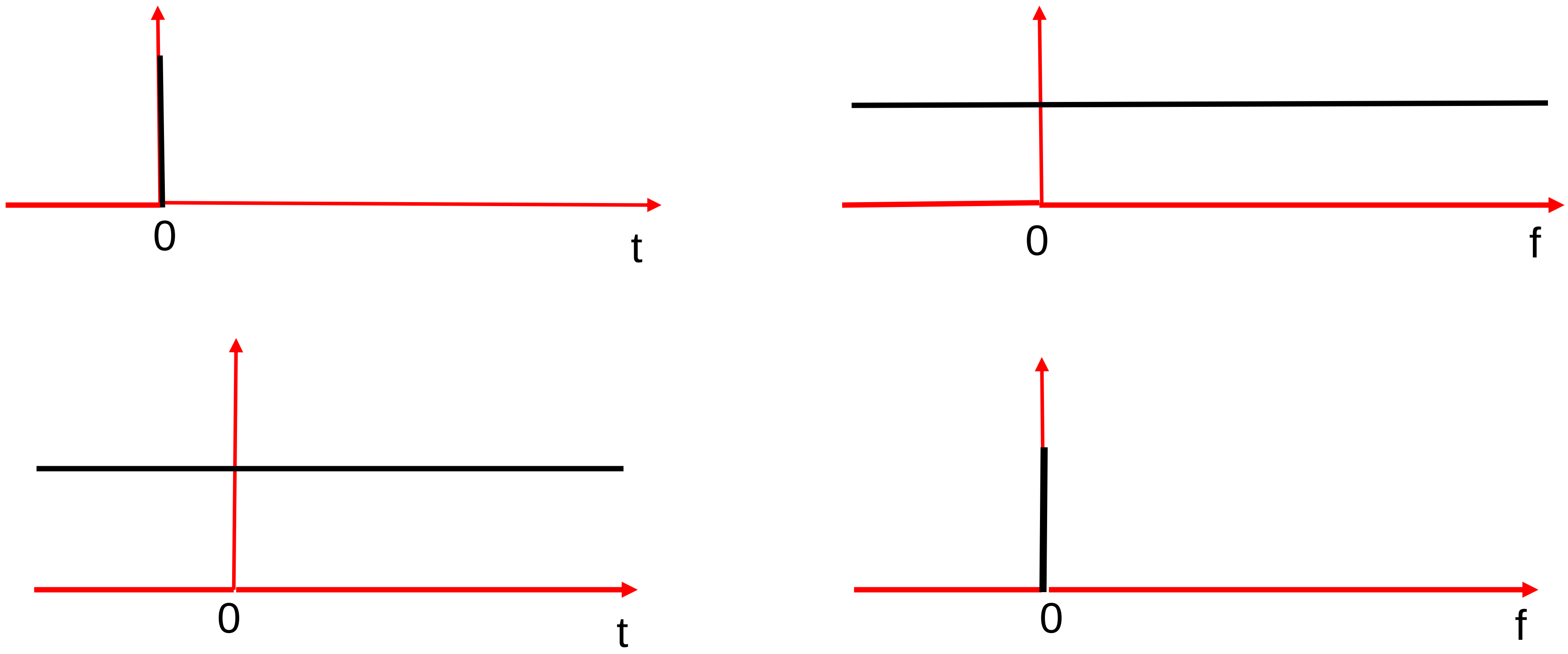
If the time signal is the convolution product of two functions, its Fourier transform is just the product of the Fourier transforms of the two signals

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \implies Y(f) = H(f)X(f)$$

If the time signal is the product two functions, its Fourier transform is just the convolution product of the two Fourier transforms

$$\begin{aligned} y(t) &= s(t)w(t) \\ Y(f) = \mathcal{F}\{s(t)w(t)\} &= \int_{-\infty}^{+\infty} S(g)W(f-g)dg \end{aligned}$$

SOME IMPORTANT PROPERTIES OF F.TRANSFORMS



$$\int_{-\infty}^{\infty} 1\cdot e^{\pm j\omega t}\,dt=\delta(\omega)$$

$$\int_{-\infty}^{\infty} \delta(t)\, e^{\pm j\omega t}\,dt=1$$

Thus, <u>frequency and time are perfect dual spaces</u>

FOURIER ANALYSIS (cont'd)

The convergence of Fourier series and Fourier transforms are assured by the Dirichlet conditions.

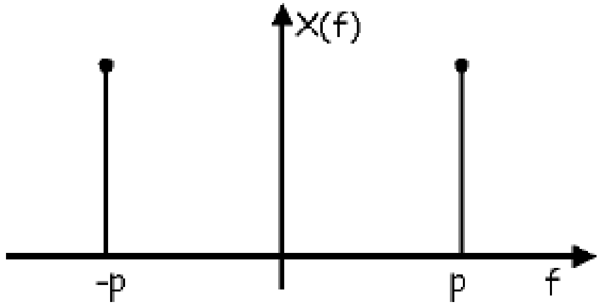
However, since the experimental data are always truncated signals, i.e. transients, only the Fourier transform is used for them.

Moreover, the introduction of Dirac δ permits to overcome the limitations provided by the Dirichlet conditions.

E.g.

$$x(t) = \cos 2\pi pt$$

$$X(f) = \frac{1}{2}\left[\delta(f+p) + \delta(f-p)\right]$$



POWER AND ENERGY SIGNALS: PARSEVAL'S THEOREMS

Assume x(t) is a periodic tension, i.e. $x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\omega_0 nt}$

The average power dissipated across a unit resistance is:

$$\frac{1}{T}\int_0^T x^2(t)dt = \sum_{n=-\infty}^{+\infty} |c_n|^2$$

Assume x(t) is a non periodic tension, i.e. $x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df$

The total energy dissipated across a unit resistance is

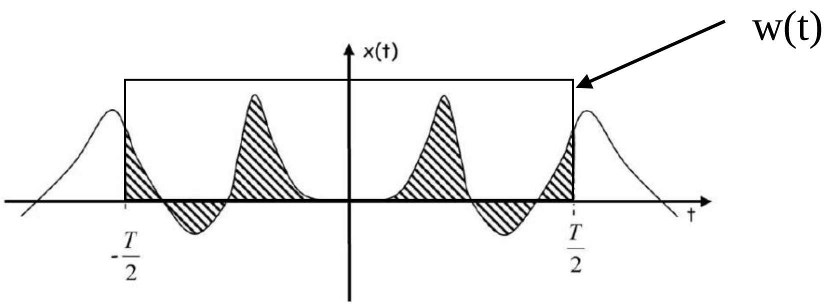
$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

Thus, periodic signals are called <u>power signals</u> while non periodic signals are called <u>energy signals</u>

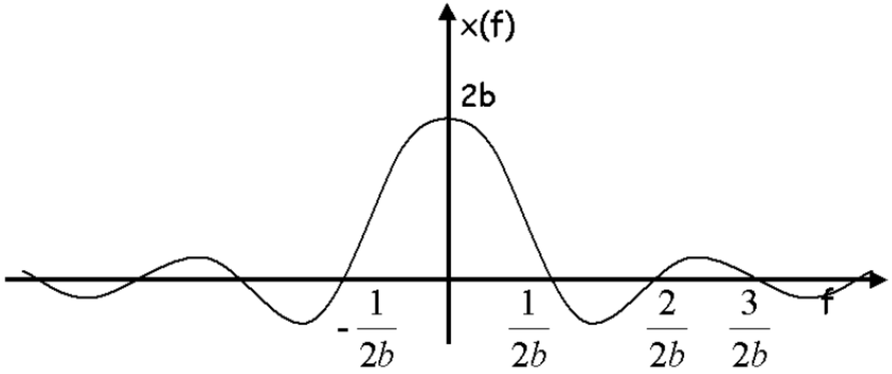
ERROR DUE TO DATA TRUNCATION: LEAKAGE

Experimental data are known only for a short period of time, e.g. between -T/2 and T/2. We can imagine they are seen through a window w(t) such that

$$\begin{cases} w(t) = 1 & |t| < T/2 \\ w(t) = 0 & |t| > T/2 \end{cases}$$



$$W(f) = \frac{T \sin(\pi f T)}{\pi f T}$$



LEAKAGE cont'd

The truncated signal is $x_T(t) = x(t)\,w(t)$ whose FT is

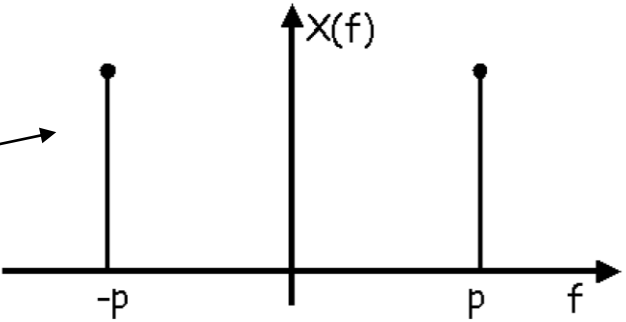
$$X_T(f) = \int_{-\infty}^{+\infty} X(g)W(f-g)dg$$

The window leads to a bias error and the convolution integral implies a distorsion on X(g), that spreads over the whole frequency range. Such distorsion is called <u>leakage</u> and, due to the sidelobes of W(f) , the X(g) frequency components spread over the frequency range

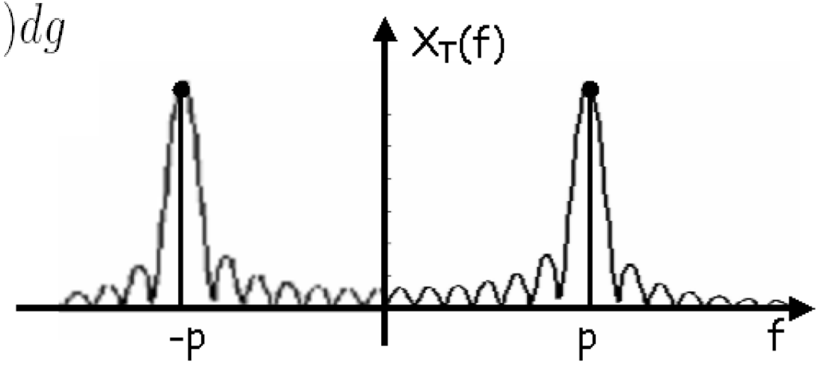
LEAKAGE (cont'd)

$$x(t) = \cos 2\pi p t$$

$$X(f) = \frac{1}{2}\left[\delta(f+p) + \delta(f-p)\right]$$

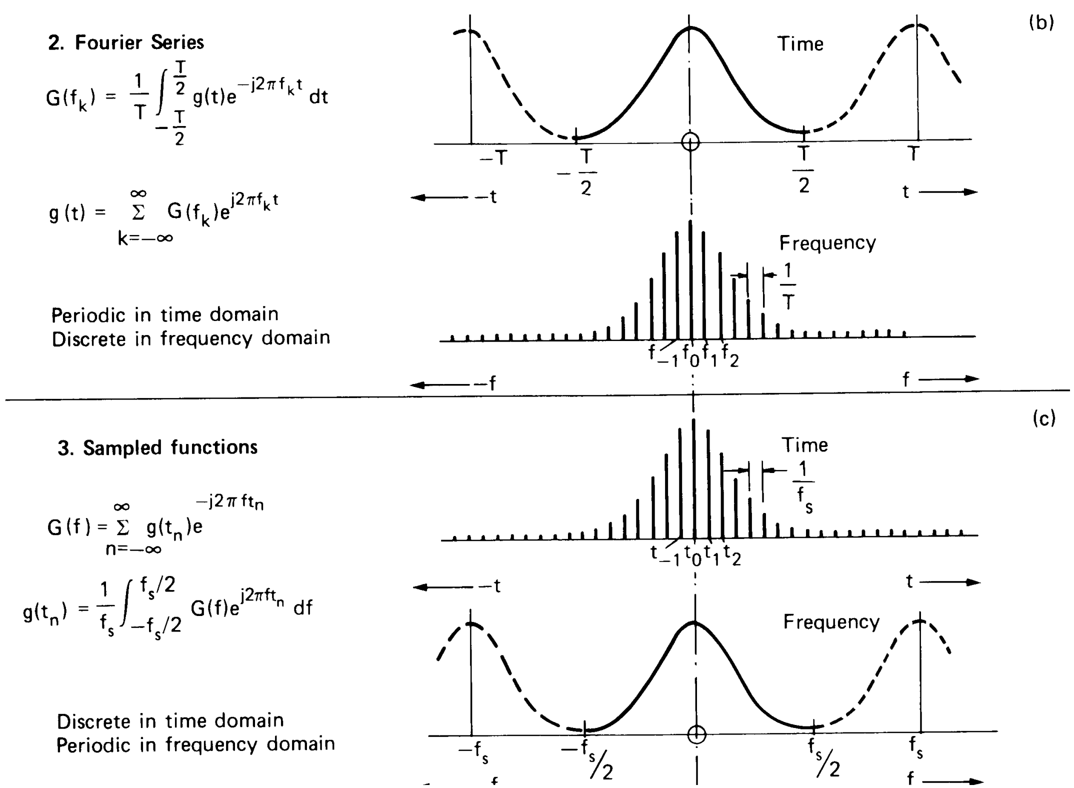


$$X_T(f) = \frac{1}{2}\int_{-\infty}^{+\infty} \left[\delta(g+p) + \delta(g-p)\right] W(f-g)\,dg$$



ALIASING

Because of the perfect duality between time and frequency domains, it is possible to depict the following relationships.

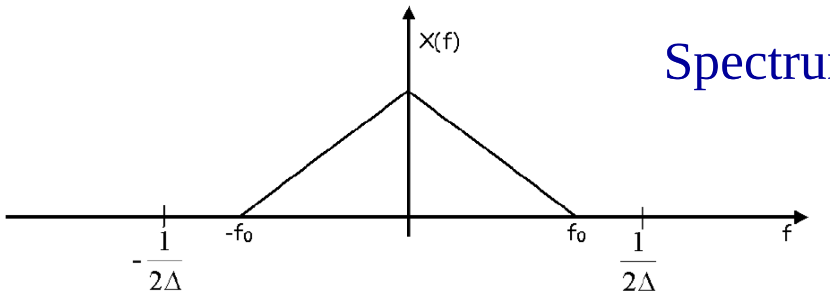


ALIASING (cont'd)

Therefore the FT of any sampled signal is periodic with period 1/Δ, i.e.

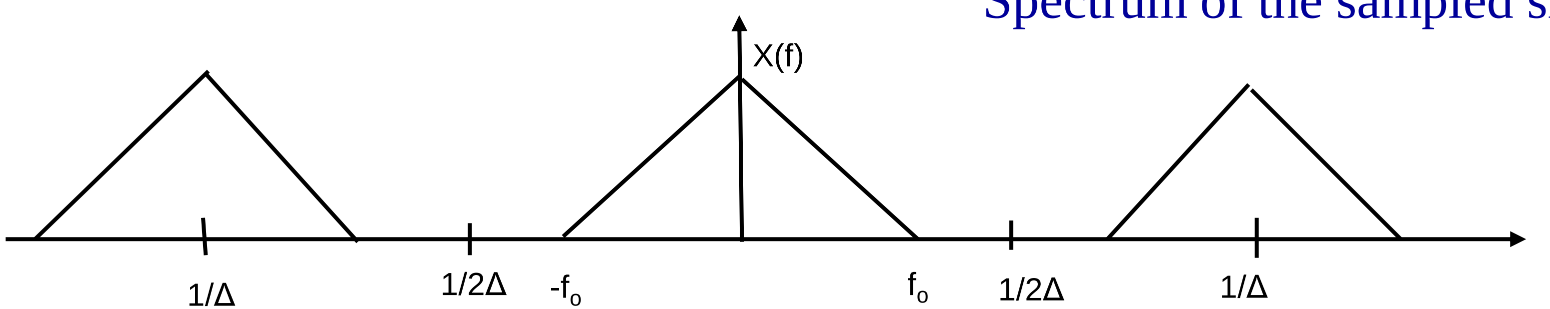
$X_s(f) = X(f + n/\Delta)$

Assuming $X(f) = 0$ for $| f | \geq \Delta/2$



Spectrum of the real signal

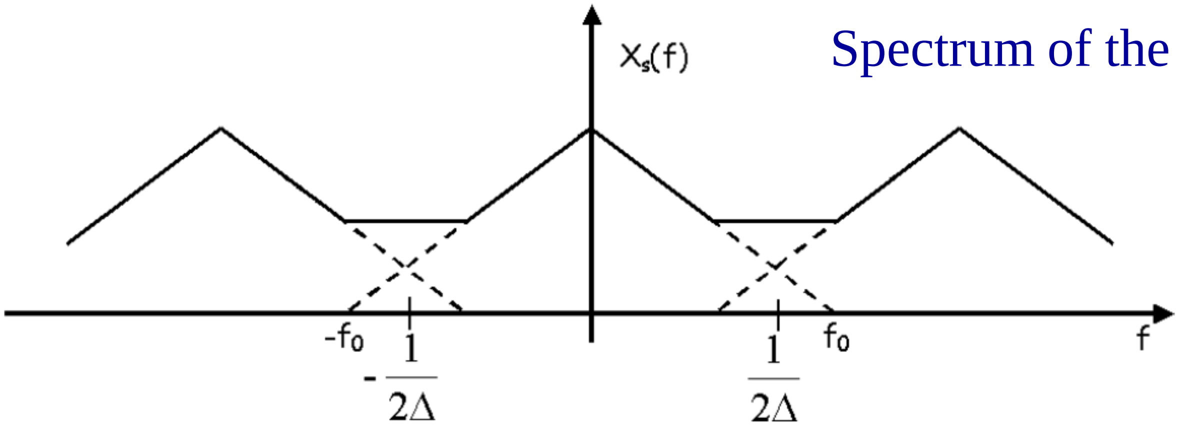
Spectrum of the sampled signal



Thus $X_s(f) = X(f)$ only for $| f | < \Delta/2$

ALIASING (cont'd)

However, if $X(f) \neq 0$ for $| f | \geq \Delta/2$



Spectrum of the aliased signal

Thus, to avoid aliasing, one must choose $1/2\Delta > f_0$ (maximum frequency of the signal)

ALIASING (cont'd)

Particularly, by calling

- $1/\Delta = f_s$ the sampling frequency
- $1/2\Delta = f_f$ the folding or Nyquist frequency
- f_0 the maximun frequency of the signal
- $T_s = 1/f_s$

To avoid aliasing one must fix $f_s > 2 f_0$

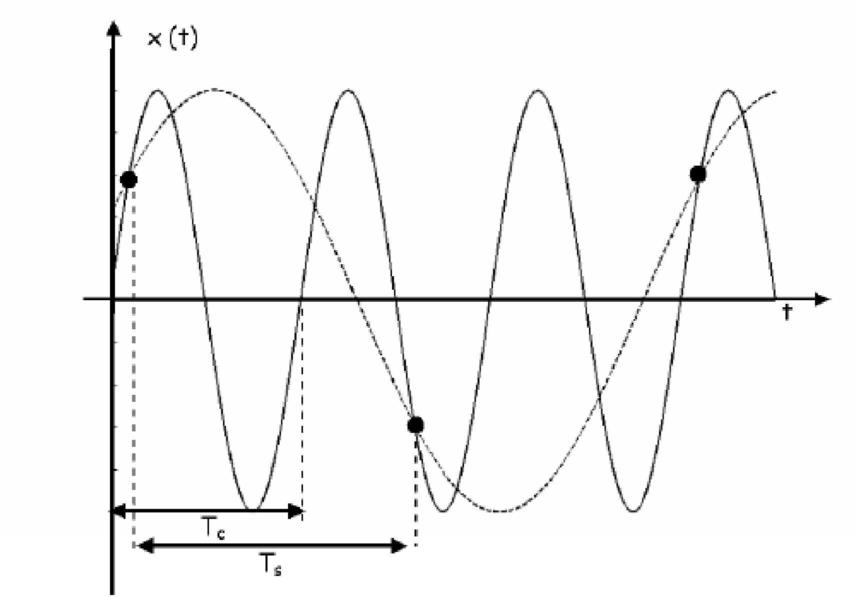
and more precisely fix the acquisition parameters as follows

$$\tau = \frac{1}{\Delta f} = \frac{N}{f_s} \leq \frac{N}{2 f_0}$$

τ = acquisition period
Δf = frequency resolution
N = number of samples

ALIASING (cont'd)

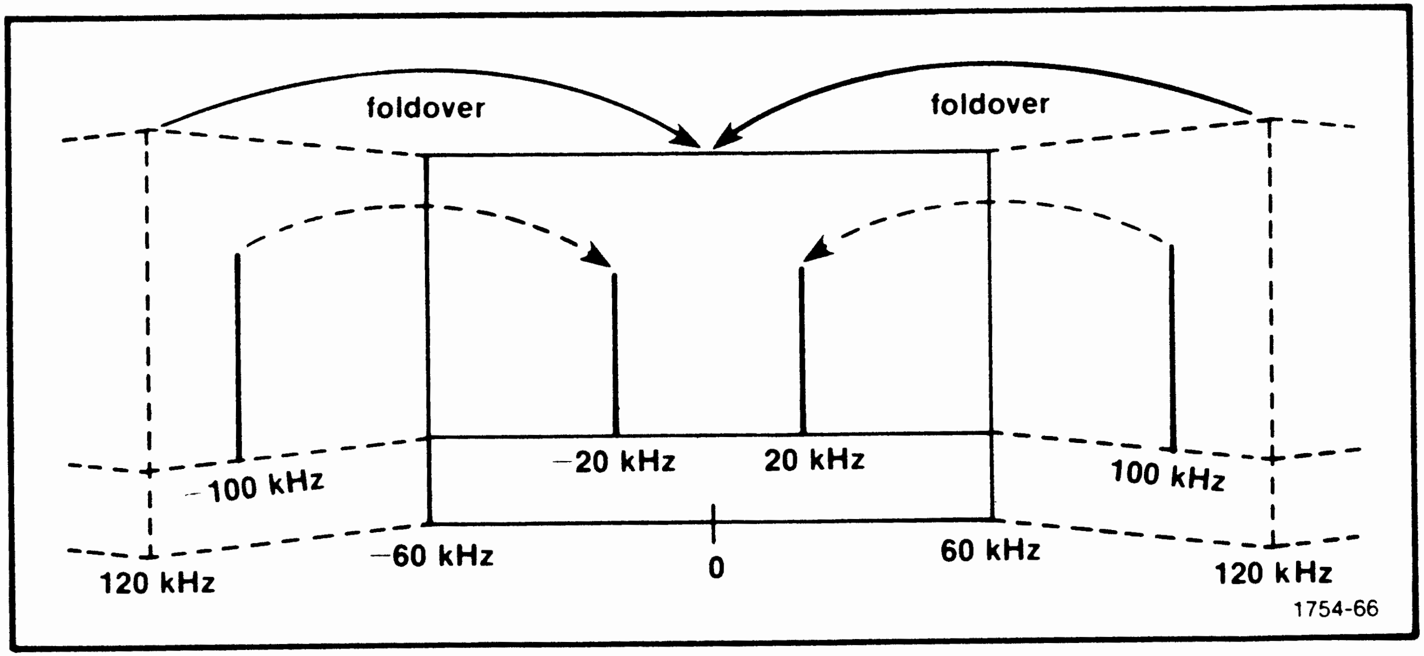
Aliasing can be also observed in the time domain



If the sampling period is larger than the half of the signal period, $T_s > T/2$, aliasing is observed

ALIASING : example

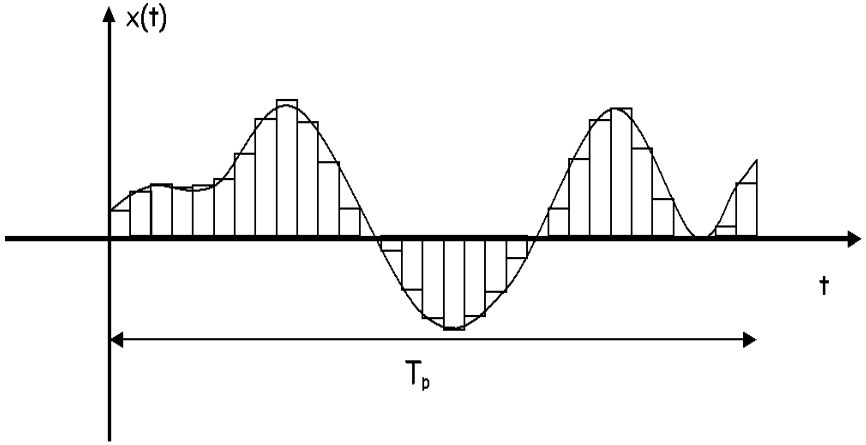
A harmonic signal (100 Hz) sampled with a sampling frequency of 120 Hz

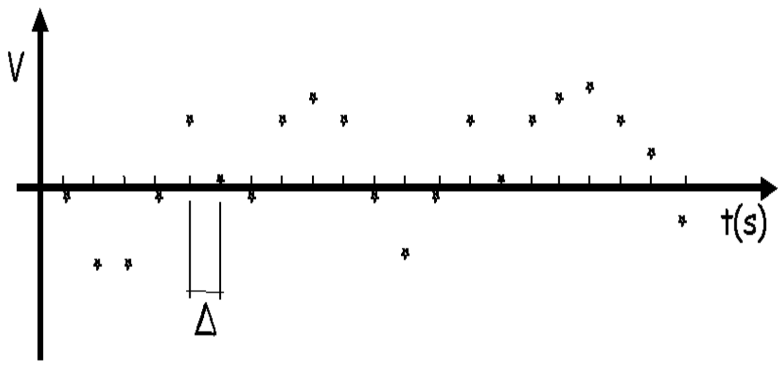


ANALOG VERSUS DIGITAL SIGNALS

Experimental data, determined by common trasducers, are generally analog signals.

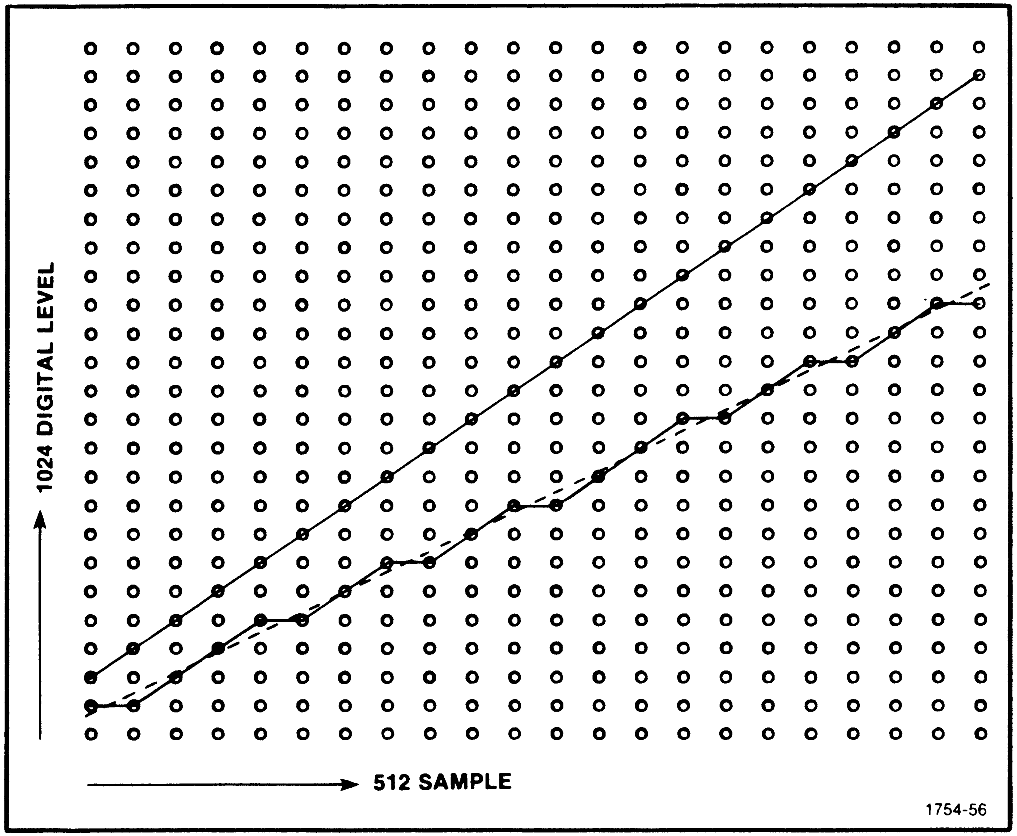
For data processing such data are always transformed into digital data by an analog to digital converter (A/D) (sampling)





$$x(n\Delta) \qquad n = 0, 1, \ldots N - 1$$

ANALOG TO DIGITAL CONVERTER

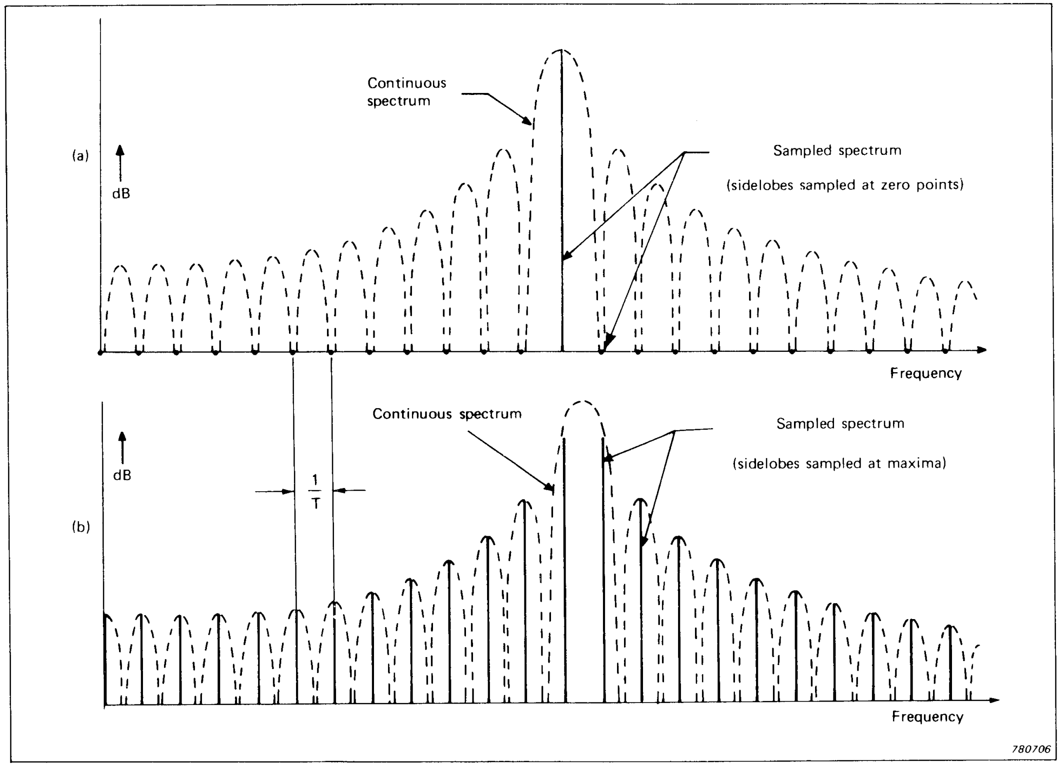


Available levels = 2^n
where n = number of bits

Depending on the number of bits of the A/D converter, one always have a a larger or smaller sampling error

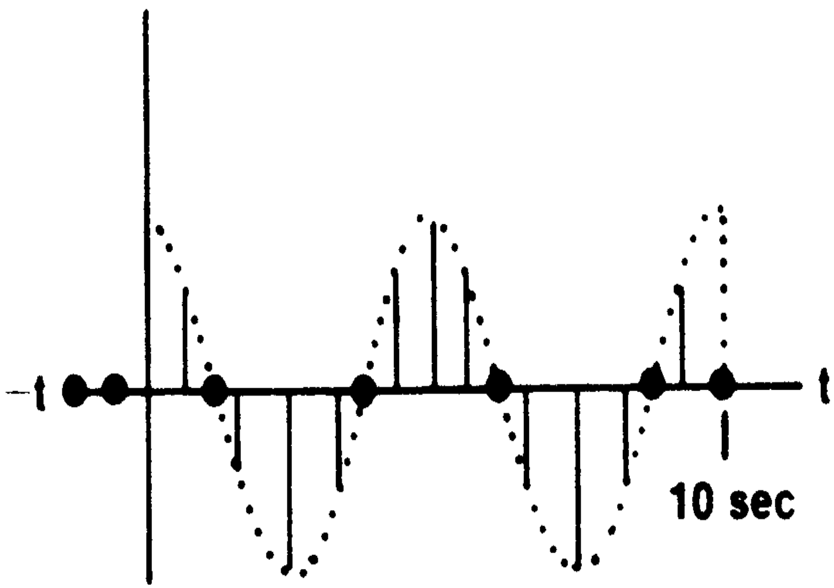
EFFECT OF SAMPLING ON LEAKAGE

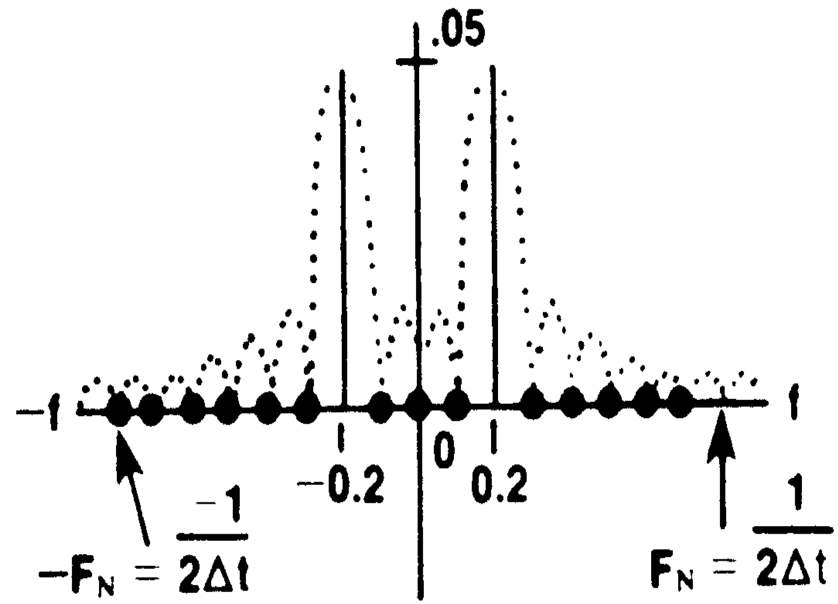
Actually, because the signal is sampled both in the time and frequency domain, we never see the sidelobes typical of the leakage error, but rather a set of sampled lines.



LEAKAGE (cont'd)

It can be shown also that, if the acquisition time corresponds exactly to the period of the periodic signal, no leakage is observed (the sampled lines exactly correspond to the zeros of the sidelobes)

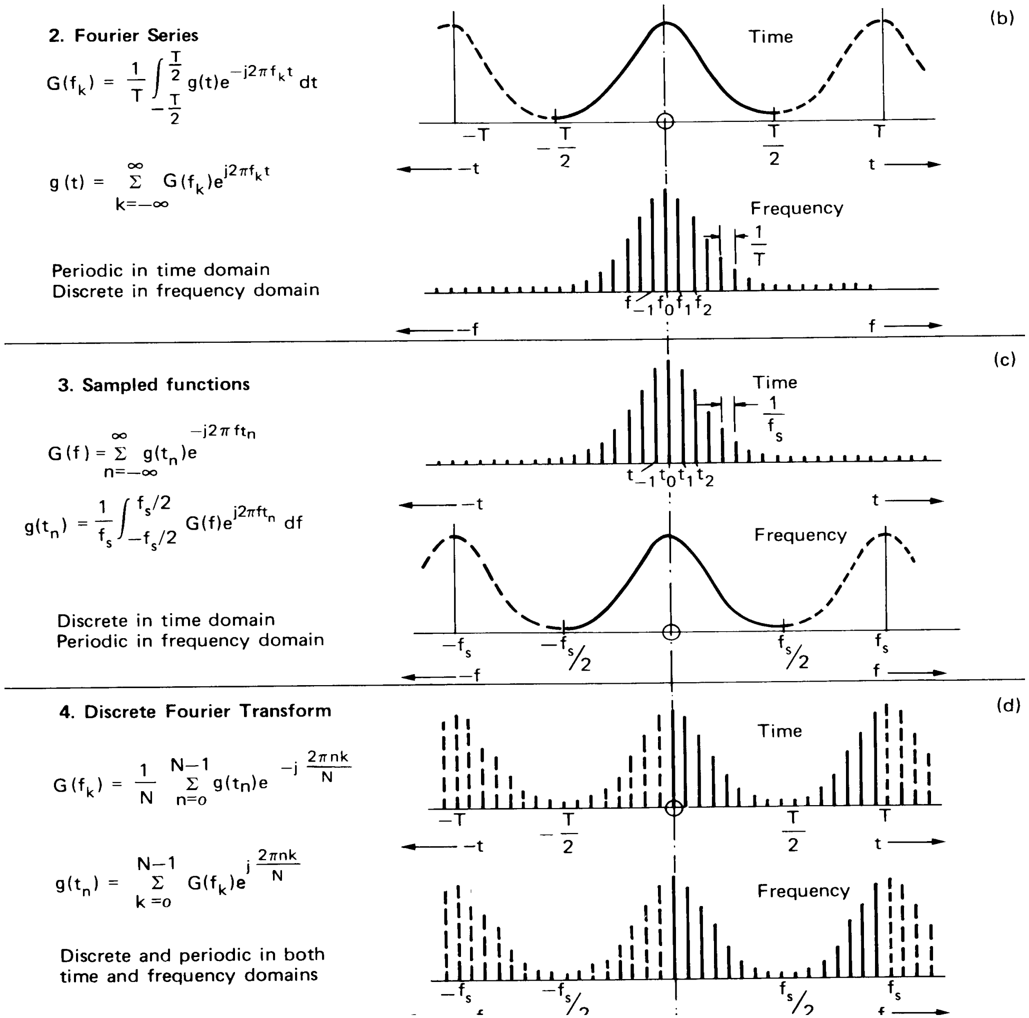




LEAKAGE (cont'd)

If T is the period

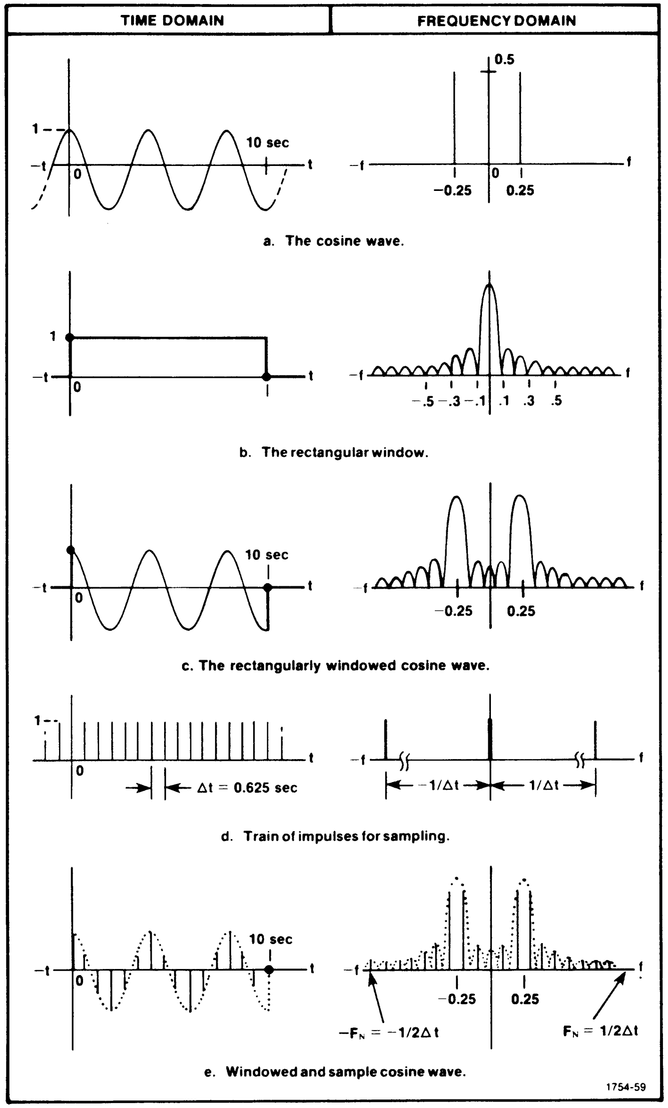
$\Delta f = 1/T$



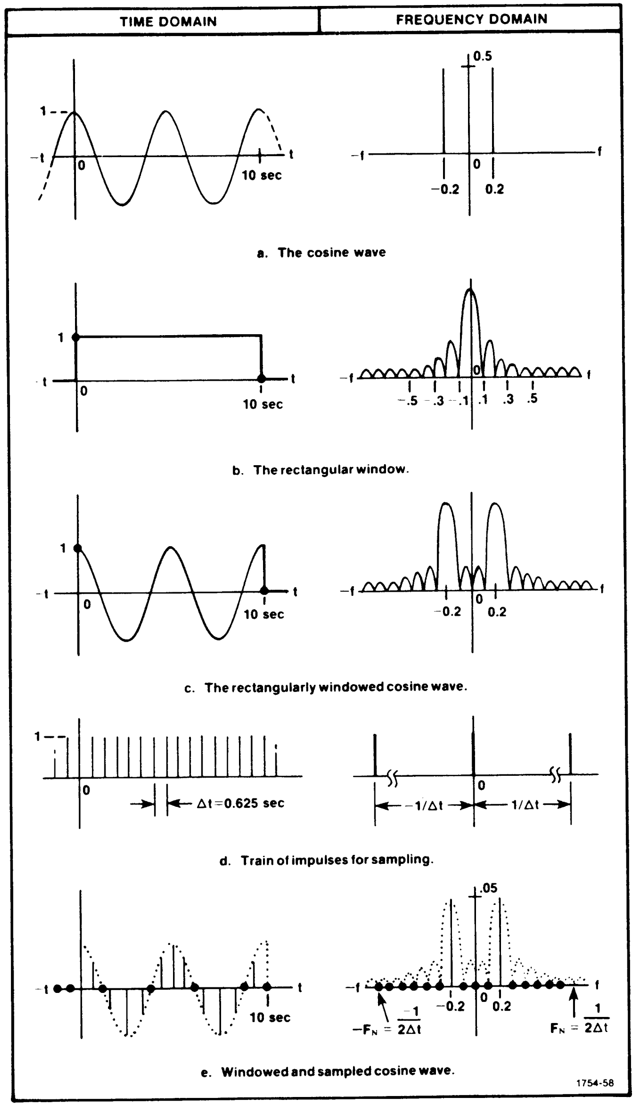
LEAKAGE (cont'd)

$$\tau = \frac{1}{\Delta f} = \frac{N}{f_s} \leq \frac{N}{2 f_0}$$

$\Delta f = 0.1$



LEAKAGE (cont'd)

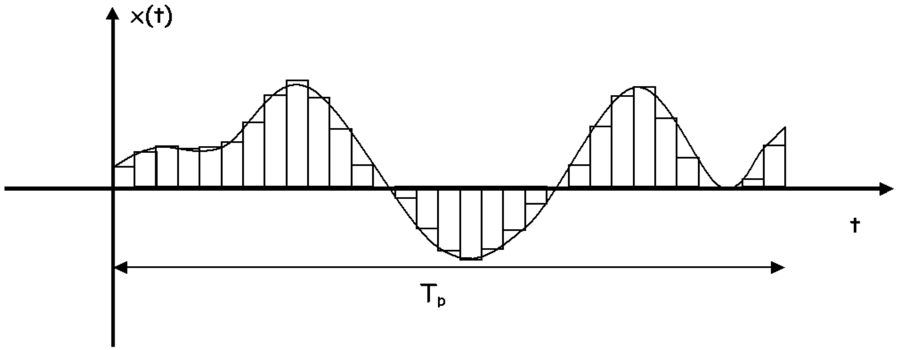


$$\tau = \frac{1}{\Delta f} = \frac{N}{f_s} \leq \frac{N}{2 f_0}$$

$\Delta f = 0.1$

DISCRETE FOURIER TRANSFORM

For sampled signals it is not possible to carry out any integral operation



It is possible however to compute a Discrete Fourier Transform that is defined as:

$$X(f)\big|_{f=\frac{k}{N\Delta}} \cong \Delta \sum_{n=0}^{N-1} x_n e^{-j\left(\frac{2\pi}{N}\right)nk}$$

N is totally arbitrary. However, for the application of the Fast Fourier Transform (FFT), a very fast algorithm of DFT, N should be chosen appropriately, i.e. $N=2^M$ where M is an integer number

CONCLUDING REMARKS

In summary there are several errors arising in signal processing.

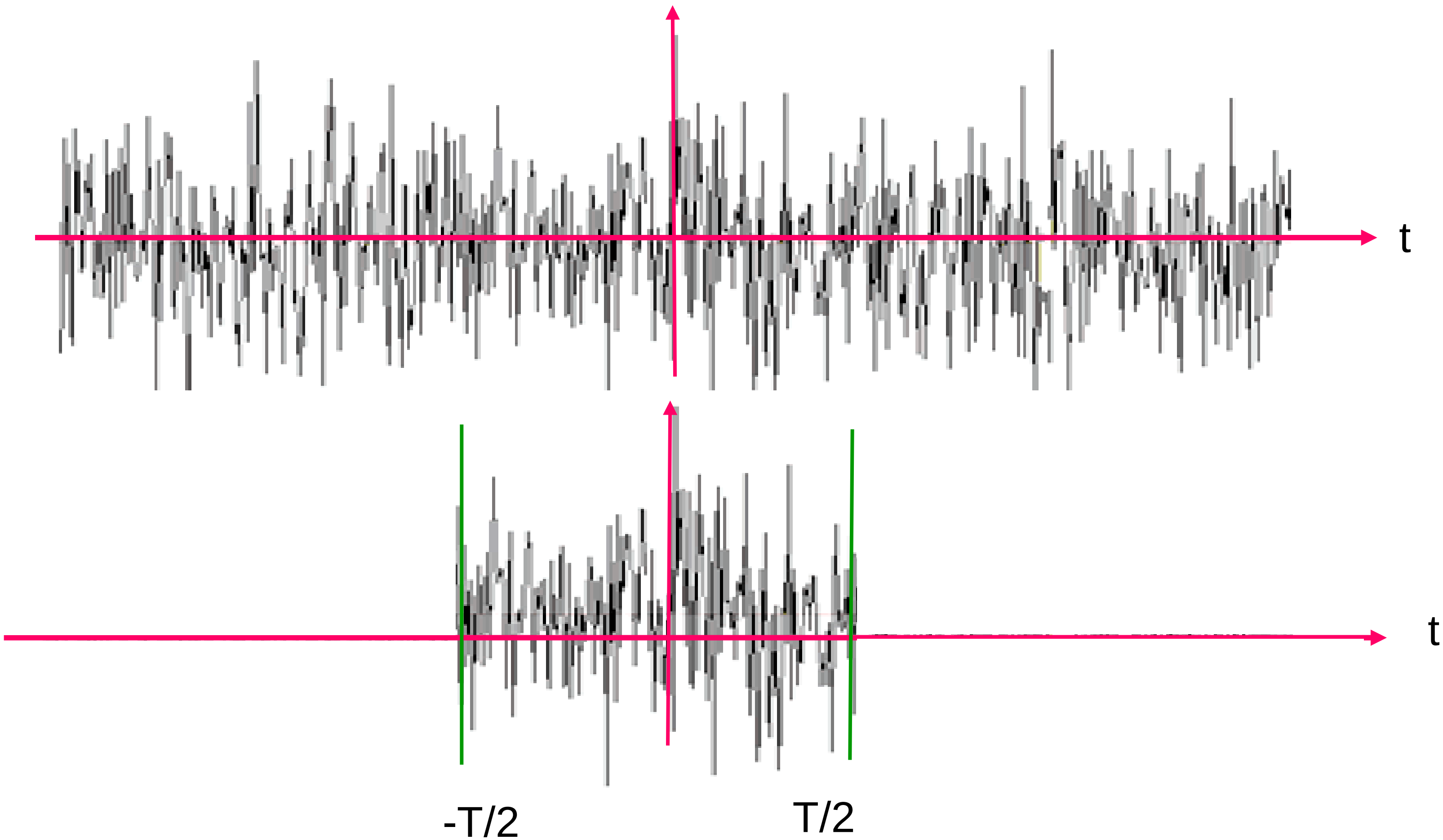
- Leakage is an error due to data truncation: it can be avoided or diminished by acquiring as large period of data as possible, or by using appropriate windows, e.g. Hanning for random data.

- Sampling errors depend on the A/C conversion and can be reduced by increasing number of bits of the A/C converter.

- Aliasing is an error due to the sampling mechanism: it can be avoided by choosing appropriately the acquisition parameters.

DESCRIPTION OF RANDOM SIGNALS IN FREQUENCY DOMAIN

Consider a sample function x(t) of a random process and a truncated version of it $x_T(t)$



DESCRIPTION OF RANDOM SIGNALS IN FREQUENCY DOMAIN

The Fourier transform of the truncated signal is

$$x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{j2\pi f t}\, d f$$

The total energy of the signal is:

$$\int_{-\infty}^{+\infty} x_T^2(t)\, d t$$

tending to ∞ for T $\longrightarrow \infty$

Thus we compute the average energy, i.e. a power

Parseval

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, d t = \frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)\, d t = \frac{1}{T}\int_{-\infty}^{+\infty} \left|X_T(f)\right|^2 d f$$

For T $\longrightarrow \infty$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)\, d t = \frac{1}{T}\int_{-\infty}^{+\infty} \lim_{T\to\infty} \left|X_T(f)\right|^2 d f$$

DESCRIPTION OF RANDOM SIGNALS IN FREQUENCY DOMAIN

The quantity $\frac{1}{T}\left|X_T(f)\right|^2$ is the spectrum of the sample. Since the first term of this equation is the average power of the sample function, the quantity

$$\lim_{T\to\infty}\frac{1}{T}\left|X_T(f)\right|^2 = S_{xx}(f)$$

is called power spectral density

Unfortunately this is not a good estimate, because it is not consistent, i.e. it does not improve by increasing T.
Thus, a different estimate is preferred. This is:

$$E\left[\lim_{T\to\infty}\frac{1}{T}\left|X_T(f)\right|^2\right] = S_{xx}(f)$$

ESTIMATE OF THE PSD (χ^2 DISTRIBUTION)

If z_1 z_2 ….. z_n are independent random variables with normal distribution, zero mean and unit variance, the new random variable:

$$\chi_n^2 = z_1^2 + z_2^2 + ... + z_n^2$$

is called chi square distribution with n degrees of freedom, mean equal to E [χ_n^2] = n and variance E[(χ_n^2- E[χ_n^2])^2] = 2n

An estimate of the PSD of a stationary process can be obtained simply by

$$\frac{1}{T}\left|X_T(f)\right|^2 = \hat{S}_{xx}(f)$$

having a frequency resolution Δf = 1/T.
X_T(f) is a complex number whose real and imaginary parts are uncorrelated random variables with zero mean and equal variance.
If x(t) is Gaussian, X_R and X_I are also Gaussian random variables (the FT is a linear operation), so that $\left|X_T(f)\right|^2 = X_R^2 + X_I^2$
is the sum of squares of 2 independent Gaussian variables.

Thus, any frequency component of the estimate $\hat{S}_{xx}(f)$ we will have a χ^2_2 distribution with 2 degrees of freedom.
Since mean and variance of a chi square distribution are n and 2n, the normalized random error (normalized standard deviation) is:

$$\varepsilon_r = \frac{\sigma[\hat{S}_{xx}(f)]}{S_{xx}(f)} = \frac{\sqrt{2n}}{n} = \sqrt{\frac{2}{n}}$$

that, for n = 2, provides ε_r = 1 (100%), i.e. the standard deviation of the estimate is as big as the quantity to estimate: unacceptable

To reduce the random error, it is possible to make m sections of period T_e from a single long random record of period T, computing for each section the FT of the signal to get $X_r(f)$, with r =1…m. This is exactly the meaning of the E operator in the expression of the PSD $E\left[\lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2\right] = S_{xx}(f)$

In this case the frequency resolution of each $X_r(f)$, is $B_e = 1/T_e$, so that

$$m = B_e/\Delta f = T/Te$$

Finally, noting that $\chi_a^2 + \chi_b^2 = \chi_{a+b}^2$, and assuming that all the sections have the same length T_e, the resultant estimate is a χ^2 variable with n=2m degrees of freedom, i.e. $n = 2 B_e T$

Substituting this value in the normalized random error, we obtain

$$\varepsilon_r = \frac{\sigma[\hat{S}_{xx}(f)]}{S_{xx}(f)} = \sqrt{\frac{2}{n}} = \frac{1}{\sqrt{B_e T}}$$

Therefore to have small normalized random errors (e.g. ε_r =0.15 or less) and obtain acceptable estimates for $S_{xx}(f)$, it is necessary to have a number of degrees of freedom close to 90, i.e. about 45 sections of the long random signal.

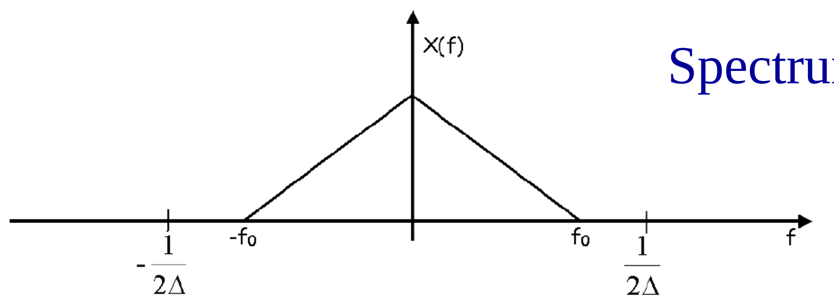
Finally, it is appropriate to have some overlap between adjacent sections: in this way we can have more averages with the same resolution, increasing the number of degrees of freedom. Usually the overlap between adjacent signals ranges between 30 and 50%.

ALIASING (cont'd)

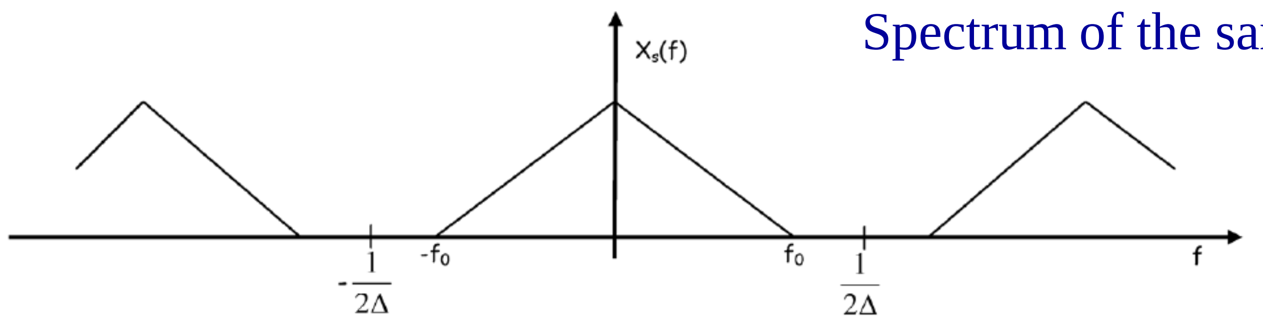
Therefore the FT of any sampled signal is periodic with period $1/\Delta$, i.e.

$X_s(f) = X(f + n/\Delta)$

Assuming $X(f) = 0$ for $| f | \geq \Delta/2$



Spectrum of the real signal



Spectrum of the sampled signal

Thus $X_s(f) = X(f)$ only for $| f | < \Delta/2$